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Gravitomagnetic and Gravitoelectric Waves in General Relativity

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Lower-order terms in expansions of the equations of General Relativity in powers of v/c (post-Newtonian approximations) have long been a source of analogies with em theory. A classic textbook example is the steadily spinning sphere generating a constant dipole gravitomagnetic field, with its associated vector potential $\mathbf{B}_0^* = \nabla \times \boldsymbol{\zeta}$ (analog of the magnetic field $\mathbf{B}$ of a spinning charged sphere). In the nonsteady case there are associated gravitoelectric fields $\mathbf{E}^* = -\boldsymbol{\zeta}_t - \nabla\phi^*$ also, where ϕ^* is the gravitational Coulomb potential. The case of a rigid sphere spun up from rest by an external (nongravitational) torque at $t = 0$ is enlightening, as it demonstrates the generation of $\mathbf{B}^*$ and $\mathbf{E}^*$ wave fields propagating outward with the velocity of light c: for large t, $\mathbf{B}^* \to \mathbf{B}_0^*$. In a coordinate system for which the metric tensor is nearly equal to the Minkowski tensor, the three-vector potential $\boldsymbol{\zeta}$ obeys an equation isomorphic to the electrodynamic equation, that is, $\square^2\boldsymbol{\zeta} = -\mu^*\mathbf{j}^*$ with $\mathbf{j}^* = -\rho\mathbf{v}$, where ρ is the mass density, $\mathbf{v}$ the three-velocity, and $\mu^* = 16\pi Gc^{-2} = 3.7 \times 10^{-26}$ mksu, G being the gravitational constant. Significantly, one can construct a *gauge invariant* four-vector potential $\mathbf{F}^* = (ic^{-1}4\phi^*, \boldsymbol{\zeta})$, obeying field equations isomorphic to Maxwell's in the *Lorentz gauge* $F^{\alpha}_{,\alpha} = 0$. The traveling transient dipole field exerts torques on matter in its path, setting up shear strains that may be measurable for very large momentum transfers, for example, between massive astronomical bodies. A rough calculation suggests that such strains are in principle observable.

1. INTRODUCTION

The classic post-Newtonian (PN) and parametrized post-Newtonian (PPN) expansions of the field equations of general relativity (GR) in powers of $v/c \ll 1$ demonstrate the existence of gravitomagnetic (GM) and gravitoelectric (GE) fields analogous to the magnetic and electric fields of em theory (Braginsky *et al.*, 1977; Forward, 1961; Thorne, 1988). We designate these by $\mathbf{B}^*$ and $\mathbf{E}^*$ respectively; as in em theory they may be described with the help of vector and scalar potentials $\boldsymbol{\zeta}$ and ϕ^*. If, as is generally the case, we may treat $\mathbf{B}^*$ and $\mathbf{E}^*$ as small, the classic weak field approximation leads to similar results (Adler and Silbergleit, 1999).

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These fields are indeed extremely small, and have thus far not been observed in the laboratory, although ingenious state-of-the-art experiments to generate Faraday and Ampère type phenomena were proposed more than 20 years ago. Using the language of em theory, such effects would correspond to near-field GE/GM energy transfers between neighboring vibrating or rotating systems (Braginsky *et al.*, 1977).

Planetary- and astrophysical-scale phenomena appear more promising. A classic model is the GM field generated by a spinning sphere of radius r_0. For an approximately Cartesian–Minkowski space, the external $\boldsymbol{\zeta}$ field is given in Weinberg's (1972) textbook in the form

$$\boldsymbol{\zeta} = (8\pi)^{-1}\mu^* r^{-3}(\mathbf{x} \times \mathbf{J}) \quad r \geq r_0 \tag{1}$$

where $\mathbf{J}$ is the sphere's angular momentum, $\mu^* = 3.7 \times 10^{-26}$ mksu and $\mathbf{x} = \mathbf{1}_r r$, r being the field point distance from the center. This "steady-state" solution, valid for constant or slowly varying $\mathbf{J}$, gives a GM field

$$\mathbf{B}^* = \boldsymbol{\nabla} \times \boldsymbol{\zeta} \tag{2}$$

filling all space for all time. For large enough $\mathbf{J}$ a spinning sphere must generate an observable $\mathbf{B}^*$ field. This should cause a gyroscope in polar orbit about the Earth to precess at a rate of 0.04 arc sec/year (normally to the orbital plane). Measurement of this precession is the primary goal of the Stanford Gravity Probe B experiment (Everitt, 1988, 1992), to be launched in the year 2002.

This paper is concerned with the time-dependent problem, that is, with the case of rapidly changing $\mathbf{J}$. We consider a stationary sphere suddenly spun-up at time $t = 0$, that is, the case $\mathbf{J} \propto 1(t)$. It is clear that, in an infinite space, the field of Eq. (1) will be established for $t \to \infty$. It is equally obvious that this field cannot fill all of space instantaneously. The establishment of such a field must therefore involve a propagation process; relativistic invariance requires the propagation to take place with the velocity of light c.

In Section 2 we briefly review the elementary GM/GE–em isomorphisms. Section 3 deals with the resulting predictions of GM/GE radiation and, specifically, with the radiation generated by a suddenly spun-up sphere, showing how it may be observed in principle. Section 4 addresses some problems raised by these results and in particular the question of angular momentum conservation. A well-known theorem states that existence of dipole gravitational radiation in gravitating systems, interacting solely through gravitational forces, would violate angular momentum conservation (Misner *et al.*, 1988). There is in em theory an analogous argument for systems of charges of identical e/m—for example, a cloud of electrons—interacting only through Coulomb forces: angular momentum conservation prohibits the generation of magnetic dipole radiation. However, application of an external emf, as in an antenna, allows one to accelerate electrons in a conductor in any desired fashion and generate dipole radiation. In the GM case the impulsive "external" torque spinning up the sphere is of nongravitational origin; furthermore it does not violate angular momentum conservation – it merely corresponds to a transfer of angular momentum, for example, to the sphere from a pair of jets or from a colliding body.

An objective of this article is to provide an elementary mathematical framework relating these arguments to standard results of GR theory and to demonstrate the possibility of dipole radiation generated by external, nongravitational sources. Another, albeit more tenuous analogy, may be helpful here: in classical linear elasticity theory waves in an infinite solid are of two types. There are compressional modes (basically monopole but which may acquire quadrupole displacement geometries, e.g., in anisotropic models) and shear modes that are intrinsically dipole fields; in homogeneous isotropic solids these modes are uncoupled, are excited by different mechanisms, and supply different kinds of information about source and medium. We are suggesting that, likewise, the quadrupole waves of classical gravitational wave theory and the dipole GM fields discussed in this article represent independent modes of the metric, corresponding to different types of source and potentially supplying different kinds of information about high energy astrophysical events.

2. GM/GE FIELDS AND EM ISOMORPHISMS

As is well known, two classic perturbation techniques, making rather different assumptions, are commonly applied to the GR field equations.

The *Weak-Field approximation* treats perturbations of the metric $g_{\alpha\beta}$ as small, that is,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad |h| \ll 1, \qquad \eta_{jk} = \delta_{jk}, \quad \eta_{00} = -1 \tag{3}$$

This yields a set of linear equations describing the propagation of gravitational waves [we use the standard conventions ($\alpha, \beta = 0, 1, 2, 3$),

$$(j, k = 1, 2, 3), x^0 = ict, v_0 = ic]$$

The *post-Newtonian approximation* also assumes a coordinate system in which the metric tensor is nearly equal to the Minkowski tensor $\eta_{\alpha\beta}$, but treats the corrections as expandable in powers of v/c.

If both v/c and $|h|$ are $\ll 1$ these approximations overlap. We shall treat models fulfilling this dual condition.

While the metrical properties of rotating systems have played an important role in the development of general relativity (Einstein, 1921; Stachel, 1980), the concepts of GM/GE fields in such systems only came to the fore in the 1960s and

1970s (see Braginsky *et al.*, 1977, for a thorough bibliography of those years). These fields appear very naturally in the lower-order terms of the PN expansions; assuming weak fields and eliminating nonlinear terms makes the corresponding equations *almost* isomorphic to the Maxwellian formalism.

Using the GR equations of motion in the PN approximation one obtains the force **f** on a unit mass moving with velocity **v** in GM/GE fields. Taking $v^2/c^2 \ll 1$ and neglecting quadratic terms the classic result is (Braginsky *et al.*, 1977; Weinberg, 1972)

$$\mathbf{f} = -\nabla\phi^* - \zeta_t + \mathbf{v} \times (\boldsymbol{\nabla} \times \zeta) \tag{4}$$

Taking

$$\mathbf{E}^* = -\zeta_t - \boldsymbol{\nabla}\phi^* \tag{5}$$

and using Eq. (2), Eq. (4) becomes isomorphic with the Lorentz force in electrodynamics for a unit charge in electric and magnetic fields $\mathbf{E}^*$ and $\mathbf{B}^*$. Equations (2) and (5) give

$$\boldsymbol{\nabla} \times \mathbf{E}^* = -\mathbf{B}^*_t \tag{6}$$

For constant or slowly varying v/c, expansion to third order yields (Weinberg, 1972)

$$\nabla^2\zeta = -\mu^*\mathbf{j}^* \tag{7}$$

with

$$\mathbf{j}^* = -\rho\mathbf{v} \tag{8}$$

$$\mu^* = 16\pi G c^{-2} \tag{9}$$

Here ρ is the mass density and, in mks units, $G = 6.67 \times 10^{-11}$ is the gravitational constant, giving $\mu^* = 3.7 \times 10^{-26}$ mksu.

Likewise for constant or slowly varying $\rho(t)$ the Newtonian potentials obey, to the lowest order

$$\nabla^2\phi^* = 4\pi G\rho \tag{10}$$

Equations (7) and (10) imply $-\mathbf{j}^* = \rho\mathbf{v} = \mu^{*-1}\nabla^2\zeta$ and $\rho = (4\pi G)^{-1}\nabla^2\phi^*$. In our approximately flat space these quantities satisfy the mass conservation condition in the form

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho\mathbf{v} = 0 \tag{11}$$

As pointed out by Weinberg (1972; note author's use of $c = 1$ units), this equation, which is *not restricted to slowly varying* $v(t)$, $\rho(t)$ yields

$$4c^{-2}\frac{\partial\phi^*}{\partial t} + \boldsymbol{\nabla} \cdot \zeta = 0, \tag{12}$$

that is, the classic analog of the Lorentz gauge of electrodynamics with its well-known factor of 4, which appears to somewhat flaw the isomorphism.

Effects of more rapid motions appear in the next order of v^2/c^2 of the PN expansions. Thus, expanding ϕ^* to order v^4/c^4 (Weinberg, 1972) and dropping nonlinear terms, we have, to second order in v/c the wave equation (see appendix)

$$\Box^2\phi^* = 4\pi G\rho \tag{13}$$

Consistently with Eq. (7) and (12) we must then have, for rapidly varying fields

$$\Box^2\boldsymbol{\zeta} = -\mu^*\mathbf{j}^* \tag{14}$$

This result can be recovered from the Weak-Field approximation (Adler and Silbergleit, 1999, 2000).

We have therefore a system of potential field equations isomorphic with those of em theory. Further examination of this isomorphism reveals the significance of the factor 4 in the "GM Lorentz gauge" of Eq. (12). Thus in electrodynamics the scalar and vector potentials (ϕ, $\mathbf{A}$) obey

$$\Box^2\phi = -\frac{\sigma}{\varepsilon} \tag{15}$$

$$\Box^2\mathbf{A} = -\mu\mathbf{j} \tag{16}$$

where σ and $\mathbf{j}$ are the charge and current densities. The conservation of charge reads

$$\frac{\partial\sigma}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \tag{17}$$

Substituting $\sigma = -\varepsilon\Box^2\phi, \mathbf{j} = -\mu^{-1}\Box^2\mathbf{A}$ with $\varepsilon\mu = c^{-2}$ gives $\Box^2(\boldsymbol{\nabla}\cdot\mathbf{A} + c^{-2}\partial\phi/\partial t) = 0$. Since ϕ and $\mathbf{A}$ must vanish at infinity, it follows that

$$c^{-2}\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = 0, \tag{18}$$

that is, the Lorentz gauge of electrodynamics expresses conservation of charge. The factor of 4 in the otherwise isomorphic Eq. (12) is due to the fundamental difference in the source terms of the Eq. (15) and (13), that is in the constants characteristic of the gravitational and electrical Coulomb laws. This difference is not trivial, as it has a bearing on the gauge invariance of the em and GM/GE fields. Thus in the em case we use a 4-vector potential $\mathbf{F} = (ic^{-1}\phi, A_k)$ to write the field equations in gauge invariant form:

$$\Box^2 F_\alpha = -\mu j_\alpha \tag{19}$$

$$F^\alpha_{,\alpha} = 0 \tag{20}$$

These equations are invariant under the transformation $F'_\alpha = F_\alpha + \psi_{,\alpha}$ providing $\nabla^2\psi = 0$. For small fields this is equivalent to a transformation

$x'^{\alpha} = x^{\alpha} + \psi(x)$ in harmonic coordinates: Eqs. (19) and (20) are thus gauge invariant.

To rewrite the GM/GE field equations in gauge invariant form we introduce $\Phi = 4\phi^*$ and construct a 4-vector potential $\mathbf{F}^* = (ic^{-1}\Phi, \zeta_k)$.

Using the 4-vector momentum density $\mathbf{j}^* = (-ic\rho, -\rho v_k)$, Eqs. (12)–(14) become

$$\Box^2 F^*_{\alpha} = -\mu^* j^*_{\alpha} \tag{21}$$

$$F^{*\alpha}_{,\alpha} = 0 \tag{22}$$

In this form the GM/GE field equations are isomorphic to those of em theory (keeping in mind, of course, that the analogy is with a special kind of electrodynamics in which (1) all charges have the same sign and (2) Coulomb forces are solely attractive).

The connection to the standard weak-field approximation (Adler and Silbergleit, 1999, 2000) can be seen as follows. The Einstein field equations are linearized to yield

$$\Box^2 h_{\alpha\beta} = -16\pi G c^{-4} S_{\alpha\beta} \tag{23}$$

$$S_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}\delta_{\alpha\beta} T, \tag{24}$$

T being the trace of the energy momentum tensor $T_{\alpha\beta}$, where, using $x^0 = ict$, $v_0 = ic$:

$$T_{jk} = \rho v_j v_k, \quad T_{0j} = i\rho c v_j, \quad T_{00} = \rho c^2, \quad (j,k = 1,2,3). \tag{25}$$

Assuming $v^2/c^2 \ll 1$ in the source term gives $S_{00} \approx \rho c^2/2$ and

$$\Box^2 h_{00} = -8\pi G c^{-2} \rho \tag{26}$$

$$\Box^2 h_{j0} = -16\pi G c^{-4} S_{0j} \tag{27}$$

Taking $\phi^* = -h_{00}c^2/2$, $\zeta_j = -ich_{j0}$ recovers Eqs. (13) and (14) with the Lorentz gauge (12). This derivation of the GM/GE potential equations is simple and direct and can be used for weak fields and nonrelativistic velocities as an alternative to the PN expansions (Adler and Silbergleit, 1999, 2000). Using $\Phi = 4\phi^*$ we recover the gauge invariant 4-vector $\mathbf{F}^*$ obeying equations (21) and (22). This derivation of the GM/GE wave theory is simpler and more direct than using the PN expansions. The latter, however, is the usual approach to GM theory and underlines best the em isomorphism.

In Eq. (23), $h_{\alpha\beta}$ is the classic gravitational wave tensor, of which only the h_{ij} quadrupole terms (Spin-2 waves) are retained as having physical significance. The 3-vector h_{0j} (Spin-1 solution) is not gauge invariant and is treated as having no absolute physical significance (Weinberg, 1972). The same holds for the h_{00} and h_{33} (scalar or Spin 0) solutions.

The column $\mathbf{H} = (h_{00}, h_{0j})$ does not define a 4-vector verifying Eq. (12): to construct a vector satisfying the Lorentz gauge we must use a different linear combination of solutions, namely $\mathbf{F}^* = -ic(2h_{00}, h_{j0})$ If, however, $\phi^*_{,t} = 0$ (e.g., spinning sphere with fixed center) the Lorentz gauge reduces to $\nabla \cdot \boldsymbol{\zeta} = 0$ and $\mathbf{H}$ obeys the Lorentz gauge. In this special case there seems to be no essential distinction between GM/GE modes and Spin-1 waves.

3. WAVE SOURCES: RAPID SPIN-UP OF RIGID SPHERE

The electrodynamic isomorphism shows that Eqs. (27) and (14) predict a transverse propagating axial $\boldsymbol{\zeta}$ vector and a dipole radiation field. This point requires comment.

Consider a system of particles whose motions are determined by Coulomb attractions. In the em case there is a well-known proof showing that, if all e_i/m_i are the same, conservation of angular momentum prohibits the radiation of dipole fields. The argument is immediately transposable to the isomorphic GM/GE case: $e^*_i/m_i \equiv 1$ for all particles and there can be no radiation. *Conservation of angular momentum prohibits the radiation of dipole gravitomagnetic waves by systems of particles interacting solely through gravitational fields* (Misner *et al.*, 1997). A self-consistent theory of gravitating matter describes, in this sense, a closed system – it does not allow for nongravitational forces and so neither does it allow for dipole radiation. Only multipole fields, for example, the quadrupole radiation of classic gravitational wave theory, are permitted. But particles can be accelerated by other means – by what we might call *external* forces such as electrical, chemical, nuclear, or other sources transfering momentum and energy to the gravitating system. The isomorphic electromagnetic case offers the perfect analog: electrons moving about in a conductor do not radiate unless driven by an applied potential, that is, by an external force. Likewise, we may assume a rigid sphere spun up at $t = 0$ (via the application of an external torque we need not specify). Neglecting stresses and assuming perfectly rigid bodies, it is thus permissible to assign a broad range of values $\mathbf{v}(t)$ to the source function (the rhs) of Eq. (14) without violating any conservation principles.

To formulate the $\boldsymbol{\zeta}$ field external to a sphere after spin-up from rest to angular momentum $\mathbf{J}$ at $t = 0$, we note that for $r > r_0$ Eq. (14) is the homogeneous equation $\Box^2\boldsymbol{\zeta} = 0$. Now if $\mathbf{C}$ is a solution of $\Box^2\mathbf{C} = 0$, so is $\nabla \times \mathbf{C}$. Taking $\mathbf{C} = -(8\pi)^{-1}\mu^*\mathbf{J}r^{-1}1(t - r/c + r_0/c)$ we find that $\boldsymbol{\zeta} = \nabla \times \mathbf{C}$ gives, for $r > r_0$:

$$\boldsymbol{\zeta} = (8\pi)^{-1}\mu^* r^{-3}(\mathbf{x} \times \mathbf{J})[1(t - r/c + r_0/c) + rc^{-1}\delta(t - r/c + r_0/c)] \quad (28)$$

This is the solution of Eq. (14) yielding Eq. (1) for $t > (r - r_0)/c$, that is, the GM field of the steadily rotating sphere is established after passage of the wavefront at $t = (r - r_0)/c$.

If the spin-up time function is $w(t)$ (with $w \to 1$ for some $t > 0$), we replace $1(t)$ and $\delta(t)$ in this solution by $w(t)$ and $w'(t)$ and the $\boldsymbol{\zeta}_{,t}$ field is

$$\boldsymbol{\zeta}_{,t} = (8\pi)^{-1}\mu^* r^{-3}(\mathbf{x} \times \mathbf{J})[w'(t - r/c + r_0/c) + rc^{-1}w''(t - r/c + r_0/c)] \quad (29)$$

In Eq. (28) the first term in parenthesis puts in place the steady field of Eq. (1), while the second term is the contribution of the (essentially dipolar) accelerations due to application of an impulsive spin-up torque.

$\boldsymbol{\zeta}_{,t}$ is not directly measurable as a matter of principle but it should be possible to observe shear strains and torques generated by gradients of $\boldsymbol{\zeta}_{,t}$ that is, by a kind of transverse tidal effect. The second term in Eq. (29) falls off like r^{-1} and is dominant in the farfield, maximum fields are in the equatorial plane normal to $\mathbf{J}$, and the corresponding torque $\boldsymbol{\tau}$ is proportional to $\boldsymbol{\nabla} \times \boldsymbol{\zeta}_{,t} = \mathbf{B}^*_{,t^-}$. For a rod of length d small compared to a wavelength joining two masses m and normal to the wavefront $\boldsymbol{\tau} = m\Delta\boldsymbol{\zeta}_{,t} \times \mathbf{d} \approx m\,(\boldsymbol{\nabla} \times \boldsymbol{\zeta}_{,t})d^2$, that is, per unit moment of inertia:

$$\boldsymbol{\tau} = -(8\pi)^{-1}\mu^* c^{-2} r^{-1}\mathbf{J}w''' \quad (30)$$

Outward energy flow is normal to the wavefront, that is, to $\boldsymbol{\zeta}_t$ and $\mathbf{B}^*$. Analogy with the em case defines the energy flux vector $\boldsymbol{\Pi}^* = \mu^{*-1}\boldsymbol{\zeta}_{,t} \times \mathbf{B}^*$. Similarly, taking a leaf from classical formulae for plasmas, it would be possible to define suitable Lagrange density functions (Tolstoy, 1973)

In principle then, such wavefields are observable either by measuring shear strains, or using relative rotations of objects aligned with the direction of propagation, or the deflection of flows in a plane normal to the wavefront. The mean strain magnitude $|h|$ is

$$|h| = \frac{\partial(T^{-1}\int_0^T \zeta\, dt)}{\partial r} = c^{-1}|\zeta| \quad (31)$$

In the far field we keep only the dominant (second) term for ζ and assume the measurements to take place in the equatorial plane normal to J, giving

$$h \approx (8\pi)^{-1}\mu^* J r^{-1} c^{-2} w'(t - r/c + r_0/c) \quad (32)$$

While the effect is clearly small, momentum transfers in highly energetic astrophysical events, such as collisions between stars accelerated by forces of nongravitational origin (e.g., supernova explosions) could generate detectable signals. It is difficult, at this stage, to develop quantitative models of astrophysical spin-ups or spin-downs—one has little idea, for instance, of what a realistic time dependence $w(t)$ would be like—but we might obtain an idea of the orders of magnitude as follows.

Assuming $w(t) = t\,T^{-1}1(t)$ for $t \leq T$, and $w(t) = 1$ for $t > T$, maximum strain occurs at $t = r/c - r_0/c + T$:

$$h_{\max} \approx (8\pi)^{-1}\mu^* J r^{-1} c^{-2} T^{-1} \tag{33}$$

Consider a neutron star of mass 2×10^{30} kg colliding off center with a large star. Assume a relative velocity of 10^6 m s^{-1} and a lever arm (off-centerdness) of 10^{11}m. Neglecting losses gives an angular momentum transfer $J = 2 \times 10^{47}$mksu. Let the transfer time be $T \approx 10^3$s. For a distance $r = 1$ pc $\approx 3 \times 10^{16}$ m, Eq. (33) yields $h \approx 10^{-16}$ –or $h \approx 10^{-19}$ for 1 kpc. Given recent progress in noise isolation and laser interferometry (LIGO will measure extensional strains $\approx 10^{-21}$ to 10^{-24}; see Abramovici *et al.*, 1992), shear strains of this magnitude may be observable. Indeed, it is conceivable that high energy events of this type could be more easily detected by their dipole fields.

Of course this simple model is overidealized, if only because other GM/GE effects also take place in the collision of two such objects. For example, the isomorphic em theory for accelerated charges shows that the rectilinear acceleration of a point mass M of momentum $\mathbf{p}$ traveling in the direction $\theta = 0$ will produce a dipolar GE field $\mathbf{E}^* \approx (4\pi)^{-1}\mu^* r^{-1}\,\mathbf{p}_{,t}\,(t - r/c)\sin\theta$. In an elastic collision of point masses of equal M the $\mathbf{p}_{,t}$ have opposite signs and the fields cancel. For two extended masses $M_1 = M_2$ of small dimensions the geometry of the resultant field will be quadrupolar. In general the masses will be neither equal, rigid, nor point-like. In view of the many uncertainties, detailed calculations at this stage would be pointless.

4. CONCLUSIONS

The case of a stationary sphere centered at the origin of a Cartesian–Minkowski system of coordinates and spun-up at $t = 0$ to a constant angular velocity illustrates how the classic steady gravitomagnetic field [Eqs. (1), (2)] is put in place by a dipole wavefield travelling outward to infinity [Eqs. (28), (29)] This demonstrates the existence of gravitational modes isomorphic to the electromagnetic waves in free space and described by a 4-vector potential F^*_α with $F^*_0 = ic^{-1}\Phi$ where $\Phi = 4\phi^*$, ϕ^* being the Newtonian gravitational potential and $F^*_j = \zeta_j$ ($j = 1,2,3$). The latter are the components of an axial vector potential isomorphic to the 3-vector potential of electrodynamics. In general the 4-vector $\mathbf{F}^*$ obeys the inhomogeneous wave equation (19), is gauge invariant, and satisfies the Lorentz gauge $F^{*\alpha}_{,\alpha} = 0$. To conform with steady-state nomenclature we called these gravitomagnetic (GM) waves and showed, in an idealized example, how to calculate shear strains associated with the passage of a transient GM wavefront. We noted order of magnitude estimates suggesting that angular momentum transfers between massive colliding objects (stars) at ranges of 1 to 10^3 parsecs could yield observable shear strains in the $|h| \approx 10^{-16}$ to 10^{-19} range.

APPENDIX

The PN expansion assumes

$$g^{00} = -1 + \overset{2}{g}{}^{00} + \overset{4}{g}{}^{00} + \cdots \tag{1A}$$

where $\overset{n}{g}{}^{00}$ indicates the term of order $(v/c)^n$. Using $g^{00}g_{00} = 1$ gives

$$g_{00} = -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + (\overset{2}{g}_{00})^2 + \cdots \tag{2A}$$

From Weinberg (1972, p. 218):

$$\nabla^2 \overset{2}{g}_{00} = -8\pi G c^{-4} \overset{0}{T}{}^{00} \tag{3A}$$

$$\nabla^2 \overset{4}{g}_{00} = c^{-2}\partial^2 \overset{2}{g}_{00}/\partial t^2 - 8\pi G c^{-4}[\overset{2}{T}{}^{00} + 2 \overset{2}{g}_{00}\overset{0}{T}{}^{00} + \overset{2}{T}{}^{ii}] + 0(\overset{2}{g}_{ij})^2 \tag{4A}$$

Writing

$$\nabla^2 g_{00} = \nabla^2 \overset{2}{g}_{00} + \nabla^2 \overset{4}{g}_{00} + \cdots \tag{5A}$$

Noting that in the $\overset{n}{g}_{00}$ equations, the source (rhs) has terms no higher than $(v/c)^{n-2}$, for a g_{00} solution good to $0(v/c)^2$ we may drop the $\nabla^2 \overset{4}{g}_{00}$ terms—providing g_{00} does not vary too rapidly with time. If we assume that field perturbations must travel with the velocity c, then any $\partial^2 g_{00}/\partial t^2$ term acquires a factor $\approx c^2$ and must be kept. The final result, good to $0(v/c)^2$, is

$$\Box^2 \overset{2}{g}_{00} = -8\pi G/c^4 \overset{0}{T}{}^{00} \tag{6A}$$

Taking $\overset{2}{g}_{00} = -2\phi^*/c^2$, $\overset{0}{T}_{00} = \rho c^2$ yields Eq. (13)

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